Delay in atomic clocks

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In the note entitled “*Another approach to relativity* ”, we saw that the hypothesis of the conservation of energy by change of Galilean reference frame leads to an absence of time dilation between reference frames. The delay observed in experiments with an atomic clock set in motion compared with a clock that has remained stationary or moved in a gravitational field can only be explained by admitting that the energy level influences the clock's rhythm.

If T is the period of a clock at rest and T' the period of the same clock set in motion at speed u, then :

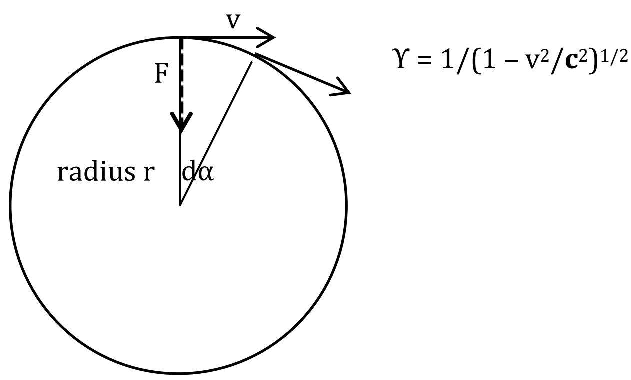
$$T'/T = E'_0/E_0 = \Upsilon = 1/(1- u^2/\mathbf{c}^2)^{1/2}$$

We will show that this result can be explained by simple reasoning.

An atomic clock is based on a quartz oscillator whose frequency is tuned to the hyperfine transition frequency of the caesium atom.

Determining the transition energy

Let's assume that the energy levels of the hyperfine transition can be modelled by circular orbits of the electron around the nucleus:



We have the relations : $v = r\, d\alpha/dt$

$dv = v\, d\alpha$ ($\overrightarrow{dv}$ is collinear with $\vec{F}$

The electron, of mass m_0 , is subject to a centripetal force $\vec{F}$. The fundamental equation of dynamics is written :

$$\Upsilon m_0\, dv/dt = F$$

or : $$\Upsilon m_0\, v^2 = r\, F$$

The force F has two components:

- one results from the electric field and is of the form A/r^2;
- the other is due to the magnetic field within the atom[1] ; the main term, linked to the orbital angular momentum, is of the form $+ Bv/r^2$ and $- Bv/r^2$ for both hyperfine levels.

As the effect of the electric field is largely predominant, we will treat the magnetic field as a perturbation superimposed on the electric field. With the electric field alone, the electron orbit is characterised by the equation :

$$\Upsilon m_0 v^2 = A/r \tag{1}$$

The addition of the magnetic field ($+ Bv/r^2$) leads to a variation in velocity Δv_1 and a variation in radius Δr_1 satisfying the equation :

$$2\,\Upsilon\, m_0\, v\, \Delta v_1 = - (A/r^2)\, \Delta r_1 + Bv/r \tag{2}$$

We neglect the variation in ϒ because we assume that the speed of the electron remains low compared with that of light.

On the other hand the overall energy variation of the electron (sum of mass + kinetic energy and electrical potential energy) is equal to the work of the force due to the magnetic field in the displacement Δr.

Knowing that: $\Delta\Upsilon_1 = \Upsilon^3\, v\, \Delta v_1\, /\mathbf{c}^2$, we can write :

$$\Upsilon^3\, m_0\, v\, \Delta v_1 + (A/r^2)\, \Delta r_1 = - (Bv/r^2)\, \Delta r_1 \tag{3}$$

Υ *being close to* 1 *and* $\Delta r/r$ *being small in front of* 1, we derive from equations (2) and (3) the relations:

$$\Upsilon\, m_0\, v\, \Delta v_1 \approx Bv/r$$

$$(A/r^2)\, \Delta r_1 \approx - Bv/r \quad \text{or :} \quad \Delta r_1 \approx - (Bv/A)\, r \tag{4}$$

Transferring the result (4) to (3), we see that the change in energy of the electron is :

$$\Delta E_1 = (B^2/A)\, v^2/r$$

The same reasoning applied to the other hyperfine level ($- Bv/r^2$) leads to :

$$\Delta E_2 = - (B^2/A)\, v^2/r$$

The transition energy between hyperfine levels is therefore :

$$\Delta E_t = \Delta E_1 - \Delta E_2 = 2\, (B^2/A)\, v^2/r \tag{5}$$

[1] The hyperfine transition corresponds to the inversion of the interaction between the nuclear magnetic dipole and the magnetic dipole linked to the orbital angular momentum and to the electron spin.

Impact of the variation in rest mass

Equation (1) can be written as :

$$(\Upsilon - 1/\Upsilon)\ m_0\,\mathbf{c}^2 = A/r$$

The overall energy of the electron is therefore :

$$\Upsilon\ m_0\,\mathbf{c}^2 - A/r = m_0\,\mathbf{c}^2/\Upsilon$$

Suppose that, as a result of the atom being set in motion or displaced in a gravitational field, the electron's rest mass is increased to the value m_0'.

This variation in mass is associated with an energy input equal to : $(m_0' - m_0)\,\mathbf{c}^2$

We can therefore write : $m_0'\,c^2/\Upsilon' - m_0\,c^2/\Upsilon = (m_0' - m_0)\,\mathbf{c}^2$

which leads to: $m_0'\,(1 - 1/\Upsilon') = m_0\,(1 - 1/\Upsilon)$

v *and* v' *being small compared with* $\mathbf{c}$, the above relationship can also be written :

$$m_0'\,v'^2 \approx m_0\,v^2 \qquad (6)$$

Let's now return to relation (5). The ratio between transition energies after and before modification of the rest mass is :

$$\Delta E_t'/\,\Delta E_t = (v'^2/r')/\,(v^2/r) = (v'^2/v^2)/(r'/r)$$

Equation (1) shows that $1/r$ varies as $m_0\,v^2$. Given (6), (r'/r) therefore remains close to 1 and the above equation becomes :

$$\Delta E_t'/\,\Delta E_t \approx v'^2/v^2 = m_0/m_0'$$

which is the desired result, since $\Delta E_t'/\,\Delta E_t$ represents the ratio of clock frequencies.